RECONSTRUCTION OF NON-STATIONARY SIGNALS WITH MISSING SAMPLES USING S-METHOD AND A GRADIENT BASED RECONSTRUCTION ALGORITHM

Miloš Brajović[*], *Ljubiša Stanković*[**] , *Miloš Daković*[***]

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Abstract: This paper addresses the reconstruction problem of non-stationary signals with missing samples. The reconstruction is achieved by using concentration measures of time-frequency representations in combination with a gradient-based iterative algorithm. As an example of time-frequency representation, the S-method is used in the proposed approach. The sparsity of the transform domain, needed for a successful reconstruction, is interpreted through the concept of concentration measures, and limits for successful reconstruction are discussed. Several examples with non-stationary signals which exhibit different concentrations in the time-frequency domain illustrate the presented theoretical concepts.

1. INTRODUCTION

Compressive sensing (CS) and sparse signals analysis have drawn a significant research attention during the last decade [1]-[11]. The CS can be applied with the assumption of signal sparsity in a transform domain. Time-frequency signal analysis can be related to CS and sparse signals analysis within several aspects, since it provides different transform domains which are sparse representations for certain classes of signals. It has been recently proposed that the S-method can be used not only for focusing ISAR radar images and in

[*] Miloš Brajović (corresponding author) is with the Faculty of Electrical Engineering, University of Montenegro, Montenegro (phone: 382-69-486639; e-mail: milosb@ac.me).

[**] Ljubiša Stanković is with the Faculty of Electrical Engineering, University of Montenegro, Montenegro (e-mail: ljubisa@ac.me).

[***] Miloš Daković is with the Faculty of Electrical Engineering, University of Montenegro, Montenegro (e-mail: milos@ac.me).

that way improving the radar signal sparsity, but also as a sparse transform directly in the CS-based recovery process, in cases when these signals have missing samples, after their elimination due to high corruption caused by noise [10]. The robust signal processing, stationary and non-stationary signal separation and reconstruction are other representative illustrations of fundamental connections between these areas [9]-[11]. The last few decades of the intensive research in the area of time-frequency analysis resulted in a large number of algorithms, methods and different mathematical approaches for the analysis and processing of non-stationary signals, with a large number of applications in several areas [12]-[18].

The area of CS deals with signals which have a certain number of missing samples. The reduced set of observations in CS is usually a consequence of a strategically chosen sampling method. On the other side, there are cases when signal samples can be intentionally omitted due to high noise corruption, or eliminated using robust techniques. All these cases can be treated as equivalent problems in the context of CS [10]. Mathematical foundation of the CS lies in fact that it is possible to reconstruct a sparse signal by interpreting the problem as an undetermined set of linear equations, which is solved by using an additional constraint – sparsity of the solution in a specific transform domain. The reconstruction process can be described as a process of finding the sparsest of all possible solutions of a given undetermined set of equations.

Although the l_0 norm represents a reasonable choice of the measure of sparsity, finding the solution by using the l_0 norm is an NP hard problem. Thus, by relaxing the reconstruction constraint and involving the l_1 norm as a measure of sparsity, it makes possible to involve different optimization approaches, such as linear programming [1], [3], [4]. Other approaches for the reconstruction include gradient-based methods, such as ones introduced in [2], [5] and [10], also recognized as suitable in the context of our problem. These methods are iterative procedures based on the gradient descent optimization.

The sparsity is a fundamental condition needed a successful CS, and the performances and outcomes of the reconstruction process highly depend on the suitable choice of the transform domain in which the signal is sparse. The basic idea behind the presented analysis lies in the fact that time-frequency representations concentrate the signal energy around the instantaneous frequency of the signal [12], [13]. Better concentration implies a smaller number of non-zero values of a time-frequency representation and thus, it can be interpreted as a sparsity measure. Concentration measures of time-frequency representations have been studied in [15], and put into the context of CS in [2] and [10]. Concentration measures of linear transforms are used in gradient-based algorithms as a measure of signal sparsity in the transformation domain [2], [5].

Within this paper, we will try investigate the possibility to relate the concept of sparsity with high level of concentration of some time-frequency representations (TFRs) via concentration measures, with the aim to reconstruct non-stationary signals with missing samples. Although a time-frequency representation can be considered as sparse for a specific class of signals, and used for its reconstruction, such as in [10] and [11], our main goal is to investigate the concept of sparsity through concentration measures for the general class of non-stationary signals, as well as limits of the reconstruction process. As an example of highly concentrated representation, the S-method, which is in detail studied in [13] and [14], is used in this paper in combination with the complex gradient reconstruction algorithm introduced in [5]. The main reason for using the gradient-based algorithm instead

of conventional compressive sensing algorithms lies in the fact that the S-method (as well as most of highly concentrated representations) as a domain of sparsity (in the context of concentration measures) has a non-linear relation with the signal, as it is stressed in [10].

2. BASIC THEORY AND PROBLEM DEFINITION

The discrete S-method, as an example of highly concentrated time-frequency representations is defined as:

$$SM_{x,L}(n,k) = \left|STFT(n,k)\right|^2 + 2\,\mathrm{Re}\left\{\sum_{i=1}^{L} STFT(n,k+i)STFT^*(n,k-i)\right\}, \tag{1}$$

where the parameter L, with typical values between 3 and 5, defines the quality (concentration) improvement of the spectrogram $\left|STFT(n,k)\right|^2$ towards the Wigner distribution, as explained in detail in [13] and [14].

A suitably chosen value of L suppresses the rise of undesired components known as the cross terms. $STFT(n,k)$ denotes the short-time Fourier transform (STFT), which is for a discrete signal $x(n)$ defined as:

$$STFT(n,k) = \sum_{m=-N_w/2}^{N_w/2-1} x(n+m)w(m)e^{-j\frac{2\pi}{N_w}km},$$

with $w(m)$ being the window function of length N_w. The discrete pseudo-Wigner distribution is defined as:

$$PWD(n,k) = \sum_{m=-N/2}^{N/2-1} w(m/2)x(n+m/2)x^*(n-m/2)e^{-4\pi mk/N}.$$

Concentration measures were studied and widely used for the optimization of time-frequency representations [13], [15]. Concentration of a TFR $\rho(n,k)$ of the signal $x(n)$ can be defined as [15]:

$$\Upsilon_\rho^p = \sum_n \sum_k \left|\rho(n,k)\right|^{1/p}. \tag{2}$$

Different values of p define different concentration measures. Highly concentrated TFRs have smaller number of non-zero values in the time-frequency plane, and thus, lower values of concentration measures. As it is known, in the context of CS the most suitable measure of sparsity, i.e. the concentration measure for counting the number of non-zero values is the l_0 norm, obtained when $p \to \infty$ in (2):

$$\Upsilon_{SM}^0 = \left\|SM_{x,L}\right\|_0 = \sum_n \sum_k \left|SM_{x,L}(n,k)\right|^0 \tag{3}$$

However, the minimization of this norm would imply a combinatorial search, which is a known NP hard problem. Additionally, since it only counts non-zero elements, even the smallest disturbances may cause problems in the minimization process [2]. This is the reason why other norms, such as l_1, are used in the CS.

According to the definitions of the S-method and the STFT [12], [13], it can be easily concluded that the l_1 equivalent norm (concentration measure) can be obtained by setting $p = 2$ in (2) as:

$$\Upsilon_{SM}^{1/2} = \left\| SM_{x,L}(n,k) \right\|_{1/2} = \sum_n \sum_k \left| SM_{x,L}(n,k) \right|^{1/2} \tag{4}$$

If the signal $x(n)$ with M_m missing samples is considered, the aim of reconstruction is to find a solution (i.e. the values of missing samples) which gives the sparsest S-method of signal. If we denote with $\mathbf{N}_x$ the set of positions of available samples, the problem of reconstruction can be formulated as:

$$\begin{gathered} \min \sum_n \sum_k \left| SM_{y,L}(n,k) \right|^{1/2} \\ \text{subject to } y(n) = x(n) \text{ for } n \in \mathbf{N}_x. \end{gathered} \tag{5}$$

where $y(n)$ denotes the reconstructed signal.

3. SOLVING THE MIINIMIZATION PROBLEM USING THE GRADIENT BASED ALGORITHM

As discussed in the Introduction, the optimization problem (5) can be successfully solved with a gradient approach. The basic idea behind the gradient based algorithm presented in [2], [5] and [10] is to set to zero the values in the signal at all missing samples positions, and then to vary these values with a small, appropriately chosen step $\pm\Delta$. Since we aim to reconstruct a complex signal, both real and imaginary parts of missing samples should be varied with the step Δ. A good starting value of the step can be obtained as $\Delta = \max \left| x(n) \right|$, $n \in \mathbf{N}_x$.

Before the procedure starts, the signal $y(n)$ with zeros at the positions of missing samples $n_i \notin \mathbf{N}_x$ is formed:

$$y(n) = \begin{cases} x(n), & \text{for } n \neq n_i \\ 0, & \text{for } n = n_i \end{cases} \tag{6}$$

Then, for each iteration k, the following steps are repeated, until the desired precision is obtained:

Step 1: For each missing sample at the position n_i, form four signals follows:

$$y_1^{(k)}(n) = \begin{cases} y^{(k)}(n) + \Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i, \end{cases}$$

$$y_2^{(k)}(n) = \begin{cases} y^{(k)}(n) - \Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i, \end{cases}$$

$$y_3^{(k)}(n) = \begin{cases} y^{(k)}(n) + j\Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i, \end{cases}$$

$$y_4^{(k)}(n) = \begin{cases} y^{(k)}(n) - j\Delta, & \text{for } n = n_i \\ y^{(k)}(n), & \text{for } n \neq n_i. \end{cases}$$

Step 2: Estimate the real and imaginary gradient parts as differences of the concentration measures:

$$g_r(n_i) = \sum_n \sum_k \left| SM_{y_1,L}(n,k) \right|^{1/2} - \sum_n \sum_k \left| SM_{y_2,L}(n,k) \right|^{1/2}, \tag{7}$$

$$g_i(n_i) = \sum_n \sum_k \left| SM_{y_3,L}(n,k) \right|^{1/2} - \sum_n \sum_k \left| SM_{y_4,L}(n,k) \right|^{1/2}. \tag{8}$$

Step 3: Form the gradient vector $\mathbf{G}^{(k)}$ of the same length as the analyzed signal $x(n)$ with elements, defined as follows:

$$G^{(k)}(n) = \begin{cases} g_r(n_i) + j g_i(n_i), & \text{for } n = n_i \\ 0, & \text{for } n \neq n_i, \end{cases} \tag{9}$$

where $g_r(n_i)$ and $g_i(n_i)$ are calculated in the Step 2.

Step 4: Correct the values of $y(n)$ using the gradient vector $\mathbf{G}^{(k)}$ with the steepest descent approach:

$$y^{(k+1)}(n) = y^{(k)}(n) - \frac{1}{NN_w} G^{(k)}(n). \tag{10}$$

As (9) is proportional to the error $y(n) - x(n)$ for both real and imaginary parts, the missing values will converge to the true signal values. In order to obtain a high level of precision, the step Δ should be decreased when the algorithm convergence slows down. In this paper the fixed step is used.

Since the values of missing samples are varied, the measure gradient enables to approach the optimal point which minimizes the measure of the S-method, meaning that the solution which gives the smallest number of non-zero values of the S-method is obtained. Until the optimal point is approached, the zero-valued or inaccurate missing samples cause a larger number of non-zero values, as it is analyzed in [7], and thus, the larger concentration measures. The satisfactory reconstruction results are obtained as long as the S-method concentration is high.

In order to explain the basic motivation for the use of the difference of concentration measures [2] consider, without loss of generality, the signal pair $y_1^{(k)}(n)$ and $y_2^{(k)}(n)$. The difference of concentration measures, which defines the gradient $G^{(k)}(n)$ used in (10), detects whether adding or subtracting Δ from the signal value decreases the concentration measure. If $y_1^{(k)}(n)$ has a higher value of the corresponding concentration measure than $y_2^{(k)}(n)$, then the difference (7) will be have a positive value, i.e. in order to obtain lower concentration (which corresponds to the increase of the sparsity), the signal value should be decreased, which is done in (10). If $y_2^{(k)}(n)$ has a higher value of the concentration measure, then the signal value should be increased in order to provide a more sparse solution. In that case, the measure difference (7) is negative, and the value $y^{(k)}(n)$ in (10) is thus increased.

4. EXAMPLES AND DISCUSSION

The presented reconstruction approach will be illustrated with three examples, where signals exhibit different concentration levels. For all examples, the corresponding S-methods will also be shown in order to illustrate the concentration level for each signal. Besides the reconstruction results, the reconstruction MSE:

$$MSE = 10\log_{10}\left(\left|y(n)-x(n)\right|^2\right) \tag{11}$$

will be also shown, in order to track the convergence of the algorithm during the iterations. In all examples, the algorithm is stopped when the number of iterations becomes equal to the length of the analyzed signal, since the obtained reconstruction error was sufficiently small. Alternative stopping criteria are presented in [2].

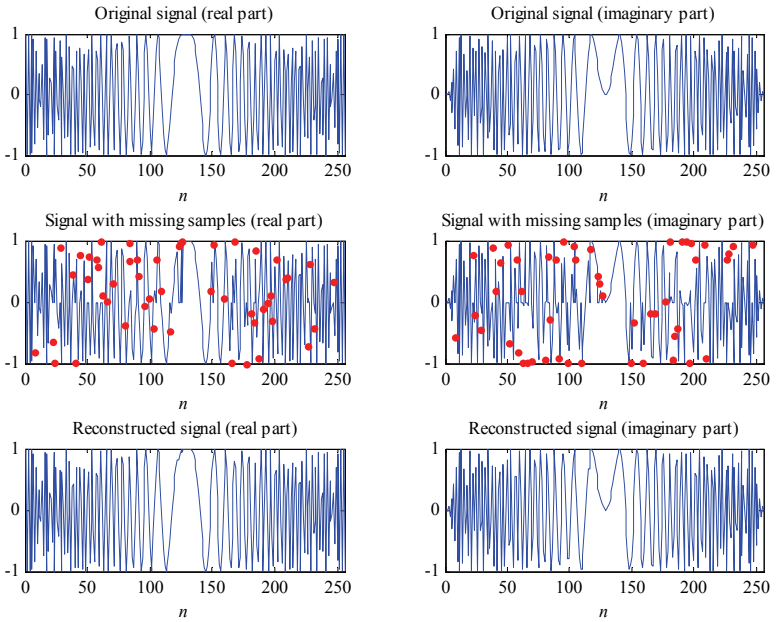

Fig. 1: The reconstruction of LFM signal with 20% of missing samples: the first row shows the real and imaginary parts of the original signal; signal with missing samples is shown in the second row, where missing values are denoted with red dots; the third row presents real and imaginary parts of the reconstructed signal.

Example 1: A mono-component linear frequency-modulated signal (LFM) of length $N = 256$, with 50 missing samples is considered:

$$x(n) = e^{j64\pi\left(\frac{n}{256}\right)^2}.$$

This non-stationary signal has very specific parameters (i.e. modulation coefficient), since its Wigner distribution achieves the highest possible concentration. Namely, the signal is located on the diagonal of the time-frequency plane, with all samples placed on the time-frequency grid. The higher order phase derivatives influence is eliminated, and thus,

this signal exhibits the highest possible concentration in the domain of Wigner distribution (in the class of non-stationary signals). Here, the S-method with $L = 100$ and a rectangular window of length $N_w = 128$ is used. This particular choice of L enables the S-method of the signal $x(n)$ to approach the concentration of the Wigner distribution.

The proposed gradient approach is successfully applied, and the reconstruction results are shown in Fig. 1. This level of signal concentration enables that the reconstruction procedure is successfully applied for 20% of missing samples at random positions. The reconstruction MSE, calculated by (11), and the S-method of the original signal are shown in Fig. 2.

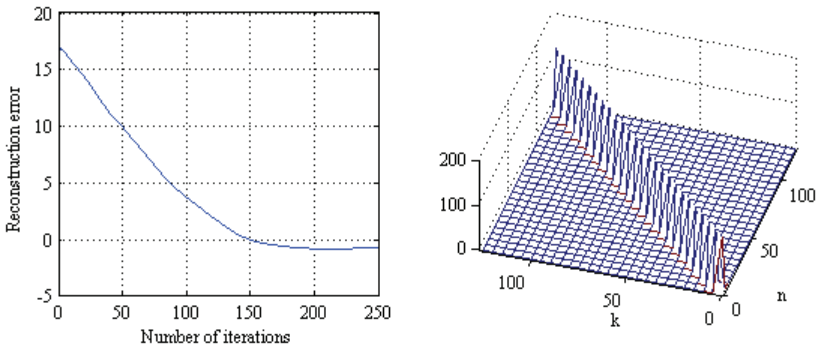

Fig. 2: The reconstruction error (left) during the iterations in the case of mono-component LFM signal from the Fig. 1 and the S-method of the analyzed signal (right).

***Example 2**:* Consider the mono-component polynomial-phase signal of the form:

$$x(n) = e^{j10\left(\frac{n+128}{128}\right)^5 - j\frac{300}{128}n},$$

and with length $N = 128$ as an example of signal with lower level of concentration. In this case, the influence of higher-order derivatives is present (since the S-method, as well as the Wigner distribution belonging to the class of quadratic time-frequency representations are ideally concentrated for signals whose phase is up to the polynomial of order 2). This influence (inner interferences) is suppressed with the Hanning window. The window length is $N_w = 32$.

The reconstruction procedure is applied on the signal realization with 14 missing samples at random positions. The reconstruction results are shown in Fig. 3. The reconstruction MSE shown in Fig. 4 indicates that the reconstruction error can be decreased with larger number of iterations. The S-method shown in Fig. 3 for the original signal (without missing samples) illustrates the signal's lower concentration. The experiment shows that the efficient reconstruction of this signal is possible for up to 11% of missing samples.

***Example 3**:* The problem of the reconstruction of a two-component signal:

$$x(n) = \exp\left(-j40\pi\left(n/128\right)^3 + j100\pi\left(n/128\right)^2 + j30n\right) + \exp\left(j50\pi\left(n/128\right)^2\right)$$

of length $N = 128$ is considered, having 10 missing samples. This signal is consisted from a linear and a quadratic FM component, thus exhibiting significantly lower concentration

than in previously studied cases. The corresponding S-method is calculated with $L = 3$ and with Hanning window of length $N_w = 64$.

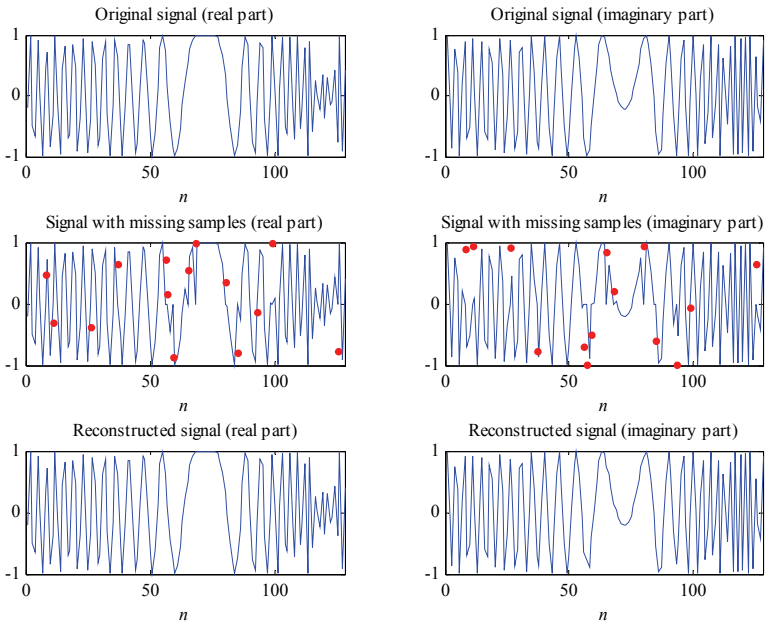

Fig. 3: The reconstruction of polynomial-phase signal with 11% of missing samples: the first row shows the real and imaginary parts of the original signal; signal with missing samples is shown in the second row, where missing values are denoted with red dots; the third row presents real and imaginary parts of the reconstructed signal

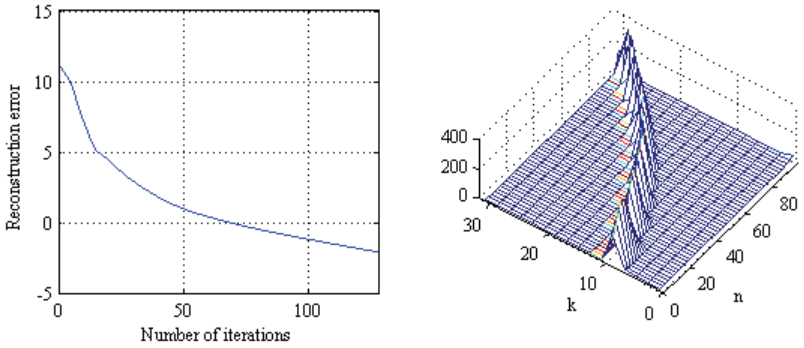

Fig. 4: The reconstruction error (left) during the iterations in the case of mono-component polynomial-phase signal shown in Fig. 3 and the corresponding S-method (right).

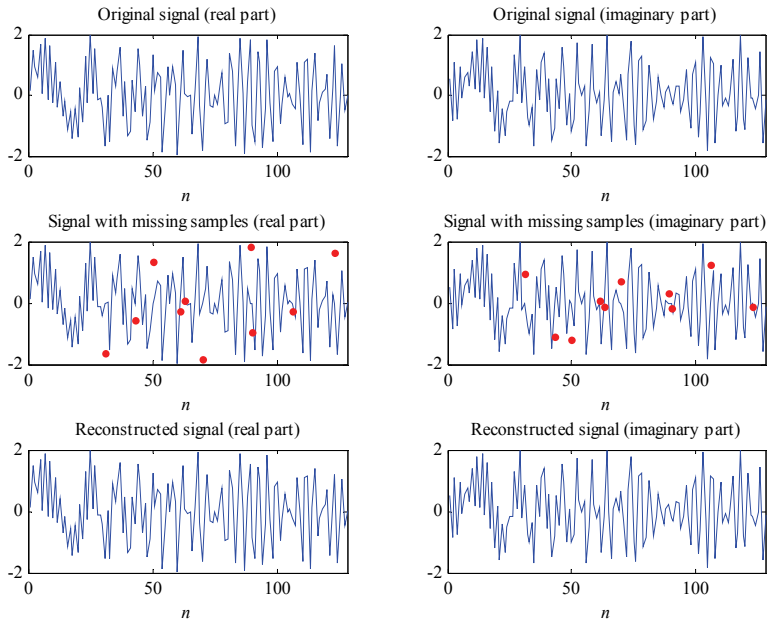

Fig. 5: The reconstruction of multi-component signal with 8% of missing samples: the first row shows the real and imaginary parts of the original signal; signal with missing samples is shown in the second row, where missing values are denoted with red dots; the third row presents real and imaginary parts of the reconstructed signal

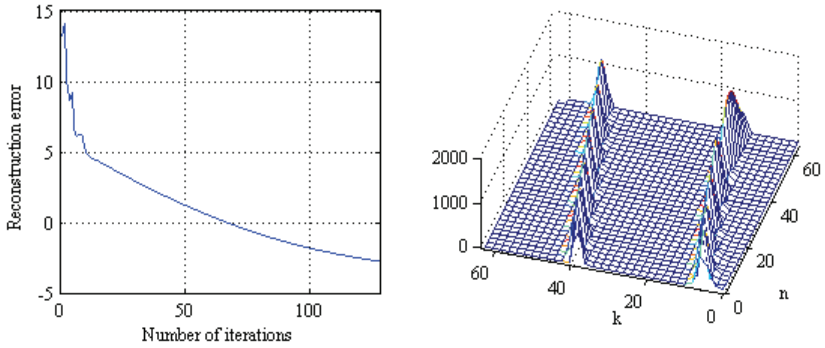

Fig. 6: The reconstruction error (left) during the iterations in the case of multi-component signal shown in Fig. 5 and the corresponding S-method (right).

Fig. 5 presents the reconstruction results, while the S-method of the analyzed signal with all samples available is shown in Fig. 6 (right), with corresponding MSE during the iterations (left). By increasing the number of algorithm's iterations, lower MSE can be achieved. This example emphasizes the fact that a suitable choice of the parameters L and N_w is significant for the successful of the reconstruction, since the S-method concentration

depends on these values. It is crucial to set the S-method parameters such that the highest possible concentration is achieved, in order to obtain a successful reconstruction.

If the signal non-stationarity is such that the sparsity property cannot be satisfied in the analyzed domain, the more concentrated time-frequency representations should be considered. The problem of the optimal time-frequency representation choice in the sense of concentration is analyzed in detail in [12] and [13]. Satisfactory results were obtained for two-component signals consisted of linear and quadratic FM components, for up to 8% of missing samples.

5. CONCLUSION AND FURTHER RESEARCH

The reconstruction of non-stationary signals using an iterative gradient-based algorithm and S-method is presented. The theoretical considerations are illustrated with three examples, by applying the reconstruction procedure on mono-component non-stationary signals with high and low level of concentration measures, as well as on a multi-component non-stationary signal. The determination of the maximal possible number of available signal samples needed for the successful reconstruction is connected with the signal sparsity, i.e. concentration measure, indicating that the reconstruction which involves highly concentrated representations may produce satisfactory results for the class of non-stationary signals, if the domain of sparsity is suitably chosen.

The extension of the presented research will move towards the analysis of the reconstruction process involving other time-frequency representations which are particularly developed in order to increase the level of signal's concentration, such as, for example, the L-Wigner distribution.

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