THE EXACT ERROR ANALYSIS IN THE INSTANTANEOUS FREQUENCY ESTIMATION BY USING QUADRATIC TIME-FREQUENCY DISTRIBUTIONS

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ABSTRACT:
General performance analysis of the instantaneous frequency (IF) estimators, for an arbitrary frequency modulated (FM) signal, is presented. Shift covariant class of quadratic time-frequency distributions as IF estimators are considered. The expressions for the IF estimator variance in the cases of white stationary and white nonstationary additive noises are derived. As special cases of this analisis, the well known results for the Wigner distribution and linear FM signal, and for the spectrogram of signals whose IF may be considered as a constant within the lag window, are presented. In addition, analysis of the linear FM signal is performed in the cases of commonly used distributions, such as spectrogram, Choi-Williams, Born-Jordan. The quite simple expression for variance of spectrogram of this signal (that is highly signal dependent) is derived. The presented expressions are checked statistically. It has been shown that the reduced interference distributions outperform the Wigner distribution, but only in the case when the IF is constant or its variations are small.

1. INTRODUCTION

Instantaneous frequency (IF) estimation is an important research topic in signal analysis [1], [2], [14]-[24], [29]-[32]. There are several approaches to the IF estimation. Time-frequency distributions (TFD) based approach is one of them [14]-[18], [20], [29]-[32]. The basis for using TFDs' in IF estimation is their first moment property, [2], [3], [10]. The first-order TFD moment, with respect to frequency, provides an acceptable IF definition for a time-varying signal. The TFD, used to recover the IF as its first moment, provides an unbiased

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estimate. The presence of noise, however, leads to a serious degradation of the first moment estimate due to the absence of any averaging in its definition. In other words, the first moment may have a high statistical variance even for high values of input signal-to-noise ratio, [24]. However, TFDs concentrate the energy of the considered signal at and around the IF in the TF plane, [2], [20], [24], [25], [29]. Consequently, as a natural alternative for the first moment, the peak detection of the TFDs' is used as an IF estimator.

The IF estimation based on TFDs maxima is analyzed in [2], [4], [14]-[18], [19], [21], [23], [24], [29]-[32]. Out of the quadratic class of distributions only the most frequently used ones are considered there: the Wigner distribution for linear frequency-modulated (FM) signal, and the spectrogram for signals with constant frequency. It has been shown that, in the case of noisy signals, this estimate highly depends on the signal to noise ratio, as well as on the window length.

In this paper we present a general analysis of an arbitrary shift covariant quadratic TFD as an IF estimator, for any frequency modulated signal. The exact expressions for the IF estimator variance in the cases of white stationary and white nonstationary noises are derived. The corresponding expressions for some frequently used TFDs from the Cohen class (CD) are obtained as special cases, as well. We presented the well known results for the Wigner distribution and linear FM signal, and for the spectrogram of signals whose IF may be considered as a constant. In addition, we have derived the variance expression for the spectrogram of a linear FM signal. This signal is considered in the cases of other commonly used TFDs, such as Born-Jordan and Choi-Williams distributions. It has been shown that the reduced interference distributions outperform the Wigner distribution, but only in the case when the IF is constant or its variations are small. For highly nonstationary signals the Wigner distribution can produce better results.

The paper is organized as follows. After this introduction the IF estimator is defined and the problem is described. In Section III the analysis of the estimation error is performed. In Section IV the variance of the estimation error in the cases of commonly used quadratic TFDs are represented. The obtained results are checked numerically and statistically in Section V.

2. BACKGROUND THEORY

Consider discrete-time observations,

$$x(nT) = f(nT) + \varepsilon(nT)\,,\;\; f(t) = A(t)\exp(j\phi(t)), \tag{1}$$

where n is an integer, T is a sampling interval, $\varepsilon(nT)$ is a white noise, and $A(t)$ is a slow varying amplitude of the analyzed signal. By definition, [5], [18], [20], [29], the IF is a first derivative of the signal phase, $\omega(t) = \phi'(t) \equiv d\phi(t)/dt$. Assume that $\omega(t)$ is an arbitrary smooth differentiable function of time with bounded derivatives $|\omega^{(r)}(t)| = |\phi^{(r+1)}(t)| \leq M_r(t)$, $r > 1$.

General form of the quadratic shift-covariant TFD's, in discrete-time domain, is defined by:

$$C_x(t,\omega;\varphi_h) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)x(t+mT+nT)x^*(t+mT-nT)e^{-j2\omega nT}, \tag{2}$$

where $\varphi_h(mT,nT)=(T/h)^2\varphi(mT/h,nT/h)$, and the time-lag kernel $\varphi(t,\tau)$ is a symmetric function in both time and lag axes. Suppose that $\varphi(t,\tau)$ has a finite length along time and lag directions, $\varphi(t,\tau)=0$, for $|t|>1/2$ or $|\tau|>1/2$. It means that $\varphi_h(mT,nT)$ has a finite length along both directions denoted by *h*, *h*>0. Note that *h* is used in definition of the CD in order to localize the estimate.

Let us analyze the CD of the signal *f*(*t*). Using the fact that the signal has a slow-varying amplitude $f(t+mT\pm nT)\varphi_h(mT,nT)\cong A(t)\exp[j\phi(t+mT\pm nT)]\varphi_h(mT,nT)$, and expanding $\phi(t+mT\pm nT)$ into the Taylor series around *t* (up to the third order therm), we get:

$$C_f(t,\omega;\varphi_h)=|A(t)|^2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)e^{-j[2(\omega-\phi'(t))nT-2\phi^{(2)}(t)mTnT-\Delta\phi(t,mT,nT)]} \quad (3)$$

where $\Delta\phi(t,mT,nT)$ is the residue of the phase which may be represented as:

$$\Delta\phi(t,mT,nT)=\sum_{s=3}^{\infty}\frac{\phi^{(s)}(t)}{s!}\sum_{k=0}^{s}\binom{s}{k}(mT)^{s-k}(nT)^k[1-(-1)^k]. \quad (4)$$

Note that TFDs from CD would have a maximum at $\omega=\phi'(t)$ if $\phi^{(s)}(t)=0$ for $s\geq 2$. The IF estimate may be defined as a solution of the following problem, [18], [24], [29]:

$$\hat{\omega}_h(t)=\arg[\max_{\omega\in Q_\omega}\{C_x(t,\omega;\varphi_h)\}], \quad (5)$$

where $Q_\omega=\{\omega:0\leq|\omega|<\pi/(2T)\}$ is a basic frequency interval. The estimation error, produced at a time-instant *t*, is:

$$\Delta\hat{\omega}_h(t)=\omega(t)-\hat{\omega}_h(t). \quad (6)$$

3. ANALYSIS OF THE ESTIMATION ERROR

Since the estimate of IF $\hat{\omega}_h(t)$ is defined by the stationary point of $C_x(t,\omega;\varphi_h)$, the $\hat{\omega}_h(t)$ is determined by zero value of $\partial C_x(t,\omega;\varphi_h)/\partial\omega$. In [12], the linearization of $\partial C_x(t,\omega;\varphi_h)/\partial\omega=0$ with respect to the small estimation error, $\Delta\hat{\omega}_h(t)$, the residual of the phase deviation, $\Delta\phi$, noise ε and squared noise ε^2, is done. There, the estimation error (6) is derived in the following general form:

$$\Delta\hat{\omega}_h(t)=\frac{1}{2R_h(t)}(P_h(t)+\frac{Q_h}{2|A(t)|^2}), \quad (7)$$

where:

$$R_h(t)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)(nT)^2e^{j2\phi^{(2)}(t)mTnT}, \quad (8)$$

$$P_h(t)\cong\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)\Delta\phi(t,mT,nT)(nT)e^{j2\phi^{(2)}(t)mTnT}, \quad (9)$$

$$Q_h = \frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_\varepsilon + \frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_{\varepsilon^2}, \tag{10}$$

$$\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_{\varepsilon^2} = 2\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)\varepsilon(t+mT+nT)\varepsilon^*(t+mT-nT)\times$$
$$\times(-jnT)e^{-j2\phi'(t)nT}, \tag{11}$$

while $|_0$ means that the preceding derivatives are calculated at the point $\omega=\phi'(t)$, $\varepsilon=0$, and $\Delta\phi(t,mT,nT)=0$.

In order to get the exact value of the IF estimator variance, the term $\partial C_x(t,\omega;\varphi_h)/\partial\omega|_0\,\delta_\varepsilon$ will be expressed by using the inner-product form of CD, [7]:

$$C_x(t,\omega;\varphi_h) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\tilde{\varphi}_h(mT,nT)[x(t+mT)e^{-j\omega mT}][x(t+nT)e^{-j\omega nT}]^*, \tag{12}$$

where $\tilde{\varphi}_h(mT,nT)=\varphi_h((m+n)T/2,(m-n)T/2)$. Consequently,

$$\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_\varepsilon = j\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\tilde{\varphi}_h(mT,nT)(n-m)T\times$$
$$\times[f(t+mT)\varepsilon^*(t+nT)+f^*(t+nT)\varepsilon(t+mT)]e^{-j\omega(m-n)T}\Big|_0. \tag{13}$$

We may conclude that for the white noise $\varepsilon(nT)$,

$$E\left\{\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_\varepsilon\right\} = E\left\{\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_{\varepsilon^2}\right\} = 0,$$

and consequently, $E\{Q_h\}=0$. Thus, the estimation variance is:

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\} = \frac{\mathrm{var}\{Q_h\}}{16|A(t)|^4|R_h(t)|^2}, \tag{14}$$

where $R_h(t)$ is defined in (8). By expanding exponential function $\exp(j2\phi^{(2)}(t)(mT)\cdot(nT))$ into a power series, $\exp(x)=\Sigma_{i=0}^{\infty}x^i/i!$, we may represent $R_h(t)$ as:

$$R_h(t) = \sum_{i=0}^{\infty}\frac{(-1)^i(2\phi^{(2)}(t))^{2i}}{2i!}B_h(2i,2i+2), \tag{15}$$

where:

$$B_h(k,l) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi_h(mT,nT)(mT)^k(nT)^l. \tag{16}$$

For a relatively small $2\phi^{(2)}(t)\ll 1$, we can write:

$$R_h(t) \cong B_h(0,2) - 2(\phi^{(2)}(t))^2 B_h(2,4). \tag{17}$$

Note that when $h\to 0$, $T\to 0$, $h/T\to\infty$, we have:

$$B_h(k,l) \to h^{k+l}b_{k,l} = h^{k+l}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\varphi(t,\tau)t^k\tau^l\,dt\,d\tau. \tag{18}$$

Now, we will derive expressions for variance, given in general by (14).

The IF estimator variance

In the sequel the nonstationary, complex-valued, white, Gaussian noise $\varepsilon(nT)$ with auto-correlation $R_{\varepsilon\varepsilon}(t+mT, t+nT) = I(t+mT)\delta(m-n)$, $I(t) \geq 0$ will be considered. The stationary noise $\varepsilon(nT)$ is obtained as a special case of the nonstationary one with $I(t) = \sigma_\varepsilon^2$.

Proposition 1: *Let* $\hat{\omega}_h(t)$ *be a solution of* (5). *For small estimation error and an FM signal* $f(t) = A(t)\exp(j\phi(t))$ *the IF estimators' variance is:*

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\} = \frac{1}{8|A(t)|^4 |R_h(t)|^2}[2C_I(t,0;|\varphi_{h_1}|^2) + C_\varsigma(0,0;\|\tilde{\Phi}_h\|)], \tag{19}$$

where $C_\varsigma(0,0,\|\tilde{\Phi}_h\|)$ *is a quadratic distribution* (*with the new kernel* $\|\tilde{\Phi}_h\| = -\|\tilde{\Psi}_h\| \cdot \|I(t)\| \cdot \|\tilde{\Psi}_h\|^*$ *of the predefined signal* $\varsigma(t) = f(t)\exp[-j(\phi'(0)t + \phi(0))]$ *at the origin of time-frequency plane, and* $\|\tilde{\Psi}_h\| = \|A_{n-m}\| .* \|\tilde{\varphi}_h\|$. *Here,* $\|\tilde{\varphi}_h\|$ *is a matrix with elements* $\tilde{\varphi}_h(mT,nT)$ *while* $\|A_{n-m}\|$ *is a matrix with elements* $A(m,n) = n-m$, *for* $m,n = 1,2,\ldots,N$ (*N represents assumed finite limits for m,n*). *The operator* $.*$ *denotes element-by-element matrix multiplication. The* $\|I(t)\| = \|I(t+nT)\delta_{m,n}\|$ *is a diagonal matrix, with* $I(t+nT)$ *being its elements. Also,* $C_I(t,0;|\varphi_{h_1}|^2)$ *represents a quadratic distribution of* $I(t)$ *with a new kernel* $|\varphi_{h_1}(mT,nT)|^2$, $\varphi_{h_1}(mT,nT) = \varphi_h(mT,nT)(nT)$.

Special case: *Linear FM signal* $f(t) = A(t)\exp(jat^2/2)$ *corrupted by the stationary, white, Gaussian noise, produces the variance*

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\} = \frac{\sigma_\varepsilon^2}{8|A(t)|^4 |R_h(t)|^2}[2\sigma_\varepsilon^2 W_h + C_f(0,0;-\|\tilde{\Psi}_h\|^2)], \tag{20}$$

where

$$W_h = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\varphi_h(mT,nT)|^2 (nT)^2. \tag{21}$$

Proof:

Starting from the properties of the Gaussian noise $\varepsilon(nT)$, [22], it may be concluded that,

$$\mathrm{var}\{Q_h\} = \mathrm{var}\left\{ \frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_\varepsilon \right\} + \mathrm{var}\left\{ \frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0 \delta_{\varepsilon^2} \right\}. \tag{22}$$

First term in (22) is highly signal and noise dependent. The second term is signal independent and time-frequency invariant for the case of stationary noise, [14]-[18], [30]. In the case of white, complex, Gaussian noise $\varepsilon(nT)$, [1], [9], [11], [18], [27], [28]-[31], the second term from (22) can be written in the following form:

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_{\varepsilon^2}\right\} = 4\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\varphi_h(m_1T,n_1T)\varphi_h^*(m_2T,n_2T)\times$$
$$\times[R_{\varepsilon\varepsilon}(t+m_1T+n_1T,t+m_1T-n_1T)R_{\varepsilon\varepsilon}^*(t+m_2T+n_2T,t+m_2T-n_2T)+ \tag{23}$$
$$+R_{\varepsilon\varepsilon}(t+m_1T+n_1T,t+m_2T+n_2T)R_{\varepsilon\varepsilon}^*(t+m_1T-n_1T,t+m_2T-n_2T)]\times$$
$$\times(n_1T)(n_2T)e^{-j2\phi'(t)(n_1-n_2)T},$$

where $R_{\varepsilon\varepsilon}(t+mT,t+nT)=E\{\varepsilon(t+mT)\varepsilon^*(t+nT)\}$ is the noise $\varepsilon(nT)$ auto-correlation function.

Special case 1: For **nonstationary, complex, white noise**, $R_{\varepsilon\varepsilon}(t+mT,t+nT)=I(t+mT)\delta(m-n)$, $I(t)\geq 0$, we get:

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_{\varepsilon^2}\right\} = 4\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi_h(mT,nT)|^2(nT)^2\times$$
$$\times I(t+mT+nT)I^*(t+mT-nT)=4C_I(t,0;|\varphi_{h_1}|^2), \tag{24}$$

where $\varphi_{h_1}(mT,nT)=\varphi_h(mT,nT)(nT)$. Thus, in this case, noise-only dependent part of variance may be represented as a quadratic distribution of $I(t)$, with the new kernel $|\varphi_{h_1}(mT,nT)|^2$.

Special case 2: For **stationary, complex, white noise**, $I(t)=\sigma_\varepsilon^2$, we have:

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_{\varepsilon^2}\right\} = 4\sigma_\varepsilon^4\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi_h(mT,nT)|^2(nT)^2=4\sigma_\varepsilon^4W_h. \tag{25}$$

Note that, as $h\to 0$, $T\to 0$, and $h/T\to\infty$, W_h is reduced to,

$$W_h\to T^2W=T^2\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|\varphi(t,\tau)|^2\tau^2\,dt\,d\tau, \tag{26}$$

where W depends on the kernel $\varphi(t,\tau)$ type only.

The first term from (22) for real and symmetric kernel $\varphi_h(mT,nT)$ may be represented as:

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_{\varepsilon}\right\} = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\tilde{\varphi}_h(m_1T,n_1T)\tilde{\varphi}_h^*(m_2T,n_2T)\times$$
$$\times(n_1-m_1)T(n_2-m_2)Te^{-j\omega(m_1-m_2)T}e^{-j\omega(n_2-n_1)T}\times[f(t+m_1T)f^*(t+m_2T)\times \tag{27}$$
$$\times R_{\varepsilon\varepsilon}^*(t+n_1T,t+n_2T)+f^*(t+n_1T)f(t+n_2T)R_{\varepsilon\varepsilon}(t+m_1T,t+m_2T)]\big|_0.$$

Applying $\tilde{\varphi}_h(m_1T,nT)=\tilde{\varphi}_h(nT,m_1T)$ and $R_{\varepsilon\varepsilon}(t+mT,t+nT)=I(t+mT)R_{\varepsilon\varepsilon}(m-n)$, $I(t)\geq 0$, we get:

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_{\varepsilon}\right\} = 2\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\tilde{\Phi}_h(m_1T,m_2T)\varsigma(m_1T)\varsigma^*(m_2T)=2C_\varsigma(0,0;\Phi_h) \tag{28}$$

where $C_{\varsigma}(0,0;\Phi_h)$ is a quadratic distribution (with the new kernel $\tilde{\Phi}_h(m_1T,m_2T)=\Phi_h((m_1+m_2)T/2,(m_1-m_2)T/2)$) of the predefined signal $\varsigma(t)$ at the origin of time-frequency (TF) plane. Note that for the linear FM signal $f(t)=A(t)\exp(jat^2/2)$, we have $\varsigma(t)=f(t)$. The general form of new kernel $\tilde{\Phi}_h(m_1T,m_2T)$ is:

$$\begin{aligned}\tilde{\Phi}_h(m_1T,m_2T)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\tilde{\varphi}_h(m_1T,n_1T)\tilde{\varphi}_h^*(m_2T,n_2T)(n_1-m_1)T(n_2-m_2)T\times\\ \times I(t+n_2T)R_{\varepsilon\varepsilon}(n_2-n_1)e^{-j\phi'(t)(n_2-n_1)T}.\end{aligned} \quad (29)$$

Special case 1: For **stationary, white, complex Gaussian noise**, we get:

$$\tilde{\Phi}_h(m_1T,m_2T)=\sigma_\varepsilon^2\sum_{n=-\infty}^{\infty}\tilde{\varphi}_h(m_1T,nT)\tilde{\varphi}_h^*(m_2T,nT)(n-m_1)(n-m_2)T^2. \quad (30)$$

For finite limits this is a matrix multiplication form,

$$\left\|\tilde{\Phi}_h\right\|=\sigma_\varepsilon^2[\left\|A_{n-m}\right\|.*\left\|\tilde{\varphi}_h\right\|]\times[\left\|A_{m-n}\right\|.*\left\|\tilde{\varphi}_h\right\|], \quad (31)$$

where $\left\|A_{n-m}\right\|$ is a matrix with elements $A(m,n)=n-m$, for $m,n=1,2,\ldots,N$. Elements of matrix $\left\|\tilde{\varphi}_h\right\|$ are $\tilde{\varphi}_h(mT,nT)$. Let us now introduce $\left\|\tilde{\Psi}_h\right\|=\left\|A_{n-m}\right\|.*\left\|\tilde{\varphi}_h\right\|$. Then, because of symmetry and realness of the kernel $\tilde{\varphi}_h(mT,nT)$, $\tilde{\varphi}_h^*(m_2T,nT)=\tilde{\varphi}_h(nT,m_2T)$, and the asymmetry of matrix $\left\|A_{n-m}\right\|$, $\left\|A_{n-m}\right\|=-\left\|A_{m-n}\right\|$, we have:

$$\left\|\tilde{\Phi}_h\right\|=-\sigma_\varepsilon^2\left\|\tilde{\Psi}_h\right\|^2. \quad (32)$$

Thus,

$$\operatorname{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0\delta_\varepsilon\right\}=2\sigma_\varepsilon^2C_{\varsigma}(0,0;-\left\|\tilde{\Psi}_h\right\|^2). \quad (33)$$

Special case 2: For **nonstationary, white, complex, Gaussian noise**, we have:

$$\begin{aligned}\tilde{\Phi}_h(m_1T,m_2T)&=\sum_{n=-\infty}^{\infty}\tilde{\varphi}_h(m_1T,nT)\tilde{\varphi}_h^*(m_2T,nT)(n-m_1)(n-m_2)T^2I(t+nT)\\ &=-\left\|\tilde{\Psi}_h\right\|\left\|I(t)\right\|\left\|\tilde{\Psi}_h\right\|^*,\end{aligned} \quad (34)$$

where $\left\|I(t)\right\|$ is described in the Proposition. Substituting eqs.(24) and (34), as well as eqs.(25) and (33) into eq.(14) proves formulas (19) and (20), respectively.

4. THE SPECIAL CASES OF QUADRATIC TFD's

The expressions for IF estimator variance in the case of any TFD from CD may be obtained as special cases of the eqs.(19)-(20).

1. **Pseudo Wigner distribution (WD)**: For this distribution $\tilde{\varphi}_h(mT,nT)=w_h(mT)\cdot\delta(m+n)w_h(nT)$, $R_h(t)=\sum_{n=-\infty}^{\infty}w_{h_1}^2(nT)\to Th\cdot\int_{-1/2}^{1/2}w^2(\tau)\tau^2 d\tau$, where $w_h(nT)$ is the real and even window function. Thus, we get:

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\}=\frac{1}{4|A(t)|^4|R_h(t)|^2}[WD_I(t,0;w_{h_2})+2WD_{I,|\varsigma|^2}(t,0;w_{h_2})], \qquad (35)$$

where $w_{h_1}(nT)=w_h(nT)(nT)$ and $w_{h_2}(nT)=w_h^2(nT)(nT)$, while $WD_{x,y}$ denotes cross-Wigner distribution. For the case of stationary white complex noise,

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\}=\frac{\sigma_\varepsilon^2}{2|A(t)|^2}(1+\frac{\sigma_\varepsilon^2}{2|A(t)|^2})W_w\frac{T}{h^3}, \qquad (36)$$

where $W_w=\int_{-1/2}^{1/2}w^4(\tau)\tau^2 d\tau\Big/(\int_{-1/2}^{1/2}w^2(\tau)\tau^2 d\tau)^2$ is the constant, dependent on window $w(\tau)$. Its values for some commonly used windows are presented in Table I. Note that for the rectangular window $w_h(nT)$ and the case of stationary, white, Gaussian noise, we get the well known expressions from [18]:

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\}=\frac{6\sigma_\varepsilon^2}{|A(t)|^2}(1+\frac{\sigma_\varepsilon^2}{2|A(t)|^2})\frac{T}{h^3}. \qquad (37)$$

Conclude that $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ is not dependent on the phase $\phi(t)$ and its derivations in the case of analyzed FM signals, i.e. $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ is constant for all values of $\phi^{(2)}(t)$ in the case of linear FM signal.

2. **Spectrogram (SPEC)**: Here we have: $\tilde{\varphi}_h(mT,nT)=w_h(mT)w_h(nT)$. In this case the two parts of variance (22) have the following forms:

$$\mathrm{var}\left\{\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0\delta_\varepsilon\right\}=2STFT_I(t,0;w_{h_1}^2)SPEC_\varsigma(0,0;w_h)+2STFT_I(t,0;w_h^2)\times$$
$$\times SPEC_\varsigma(0,0;w_{h_1})-4STFT_I(t,0;w_{h_2})\,\mathrm{Re}[STFT_\varsigma(0,0;w_h)STFT_\varsigma^*(0,0;w_{h_1})], \qquad (38)$$

$$\mathrm{var}\left\{\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\bigg|_0\delta_{\varepsilon^2}\right\}=2\,\mathrm{Re}[STFT_I(t,0;w_{h_1}^2)STFT_I^*(t,0;w_h^2)]-2SPEC_I(t,0;w_{h_2}), \qquad (39)$$

where $STFT(t,\omega;w_h)$ represents the short-time Fourier transform, $SPEC(t,\omega;w_h)=|STFT(t,\omega;w_h)|^2$, whereas the $R_h(t)$, eq.(15)-(16), is:

$$R_h(t)=\frac{h^2}{4}\sum_{i=0}^{\infty}\frac{(-1)^i(h^2\phi^{(2)}(t)/2)^{2i}}{(2i)!}\sum_{i_1=0}^{2i}\sum_{i_2=0}^{2i+2}\binom{2i}{i_1}(-1)^{i_2}\binom{2i+2}{i_2}M_{4i+2-i_1-i_2}^w\cdot M_{i_1+i_2}^w \qquad (40)$$

and

$$M_r^w=\int_{-1/2}^{1/2}w(\tau)\tau^r d\tau \qquad (41)$$

Table I. The coeficients W_w, S_w, and C_w for different window $w(\tau)$ forms

Window $w(\tau)$	Rectangular	Hanning	Hamming	Triangular
W_w	12	54.4631	41.6581	34.2857
S_w	12	28.1135	19.7324	19.2
C_w	$5.3\cdot 10^{-3}$	$1.768\cdot 10^{-3}$	$2.936\cdot 10^{-3}$	$2.74\cdot 10^{-3}$

is the *r*-th moment of window $w(\tau)$. Substitution of eqs.(38)-(39) into eq.(22) produces variance $\mathrm{var}\{Q_h\}$. After that, substitution of the obtained variance and eq.(40) into (14) gives the IF estimator variance in the case of SPEC. From eq.(38) it can be easily concluded that the $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ in the case of SPEC is highly signal dependent.

Linear FM signal, $f(t)=A(t)\exp(jat^2/2)$, *corrupted by the stationary* (*or quasi-stationary* $I(t+nT)=I(t)$), *complex, white, Gaussian noise*: In this case we have $SPEC_\varsigma(0,0;w_{h_1})=0$ and $STFT_I(t,0;w_{h_2})=0$. Thus,

$$\mathrm{var}\left\{\left.\frac{\partial C_x(t,\omega;\varphi_h)}{\partial\omega}\right|_0 \delta_\varepsilon\right\}=2\sigma_\varepsilon^2 Th\cdot\left(\int_{-1/2}^{1/2} w^2(\tau)\tau^2 d\tau\right)\cdot SPEC_f(0,0;w_h), \tag{42}$$

where:

$$SPEC_f(0,0;w_h)=\left|A(t)\right|^2\left|\sum_{i=0}^{\infty}\frac{(jh^2a/2)^i}{i!}M_{2i}^w\right|^2 \tag{43}$$

while $R_h(t)$ is given by eq.(40) with $\phi^{(2)}(t)=a$. Now, the exact IF estimation error $\Delta\hat{\omega}_h(t)$ variance may be easily obtained by replacing eqs.(40) and (42)-(43) into expression (14).

Since *r*-th moment of the window $w(\tau)$ is very small for $r>5$, then for relatively small *a*, $a\le 0.6$, $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ can be closely approximated by the following very simple form (obtained by replacing $i=0,1$ into (40) and (43)):

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\}\cong\frac{\sigma_\varepsilon^2}{2\left|A(t)\right|^2}\frac{T}{h^3}S_w e^{a^2C_w h^4}, \tag{44}$$

where $S_w=\int_{-1/2}^{1/2}w^2(\tau)\tau^2 d\tau\Big/(M_2^w)^2$ and

$$C_w=\frac{1}{4}\left((\frac{M_2^w}{M_0^w})^2+\frac{M_6^w}{M_2^w}-\frac{M_4^w}{M_0^w}\right) \tag{45}$$

are the window $w(\tau)$ dependent constants, Table I. Note that, due to the kernel $\tilde{\varphi}_h(mT,nT)$ symmetry, the same values of variance $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ hold for negative *a* with $a\to|a|$. Conclude that in this case $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ is not constant. It is highly signal dependent. As *a* increases, $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ increases from the value

$$\mathrm{var}\{\Delta\hat{\omega}_h(t)\} \cong \frac{\sigma_\varepsilon^2}{2|A(t)|^2} S_w \frac{T}{h^3}, \text{ for } \quad a = 0 \tag{46}$$

that is derived in literature as the spectrogram variance, [17]. Of course, it holds only for $a = 0$, while for other values of *a* the more general relation (40)-(44) derived in this paper holds.

3. **Smoothed pseudo WD (SWD)**: In this case we have, [3], [5]: $\varphi_h(mT,nT) = \gamma \cdot \exp(-(mT)^2/\alpha - (nT)^2/\beta)$. For $\alpha = \beta$, $\tilde{\varphi}_h(mT,nT) = w_h(mT)w_h(nT)$, where $w_h(mT) = \sqrt{\gamma}\exp(-(mT)^2/(2\alpha))$ is the Gaussian window. Consequently, the variance expression may be directly obtained from those in the case of spectrogram, for Gaussian window $w_h(mT)$.

5. NUMERICAL IMPLEMENTATION

Obtained results for variance are checked statistically and presented in Figs.1*a*)-*c*). The following quadratic TFDs are considered:

pseudo WD, with the Hanning window $w(\tau)$;

spectrogram;

Born-Jordan (BJD), $\varphi_h(mT,nT) = \frac{1}{2|nT|+1}\mathrm{rect}\left|\frac{mT}{2nT}\right|$;

Choi-Williams distribution (CWD),

$$\varphi_h(mT,nT) = \frac{\sigma}{2\sqrt{\pi}} \cdot \frac{1}{|nT|+1}\exp[-(\frac{\sigma \cdot mT}{2nT})^2], \ \sigma = \sqrt{2}\pi .$$

The general expression (20) for variance is used in the numerical analysis. Linear FM signal $f(t) = \exp(-j16\pi a t^2)$ corrupted by the stationary noise with variance $\sigma_\varepsilon = 0.25$ is analyzed. The values of $\phi^{(2)}(t) = a$ with $a \in [0,1]$ are considered in the case of spectrogram, while $a \in [0,0.5]$ in the case of other TFDs, when the oversampling is necessary. The signal is considered within the time interval $t \in [-2,2]$ with the sampling period $T = 1/64$. The symmetric kernels, $-h/2 \le (mT),(nT) \le h/2$, with $h = 1$ width (i.e. 64 samples kernel width) are used. Note that the results for the CWD highly dependent on the parameter σ. Thus, any comparison is relative. Here we have chosen the parameters according to the results from [25].

A very high agreement of the theoretical results (thick line) and the statistical data (thin line) can easily be noted from Fig.1. Theoretical values are produced by the derived expressions (19)-(20), while the statistical data are obtained by running 128 simulations. Typical error functions for one realization are given in Fig.2. Note that $\mathrm{var}\{\Delta\hat{\omega}_h(t)\}$ in BJD and CWD cases increases (as in the case of SPEC) as *a* increases. For small $a \to 0$ they have lower variance than the PWD, while by increasing *a* they perform worse than the PWD. These conclusions are expected since the RID distributions significantly reduce no-

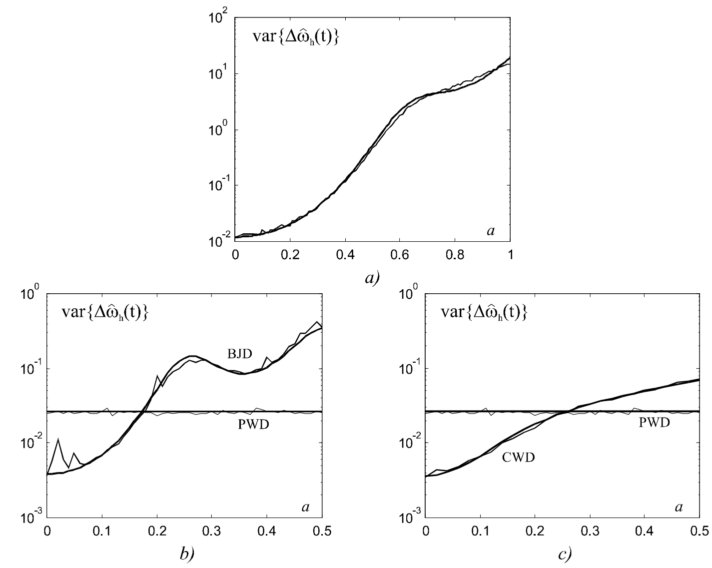

Fig.1. IF variance obtained theoretically (thick line) and statistically (thin line) for different normalized values of $\phi^{(2)}(t) = a$; *a*) SPEC, *b*) BJD and pseudo WD, *c*) CWD and pseudo WD. Note that $a = 0$ corresponds to the pure sinusoid, while value of *a* at the ending interval point corresponds to the diagonal in the considered time-frequency domain.

ise energy located far from the θ,τ axes. For the signals whose ambiguity function lies along the θ,τ axes (as in the case of linear FM signals with $a \rightarrow 0$) the RID distributions do not degrade signal representation. On the other hand, for linear FM signals with larger values of *a*, the RID distributions significantly degrade representation of the analyzed signal. Consequently, in this case it may happen that the TFDs from RID class have worse performance than the WD. A decrease in variance for the BJD, for *a* between 0.3 and 0.4, is due to its pseudo form. Namely, considering finite support of the BJD a significant kernel values can be truncated since they are the θ-τ domain oscillatory. They can cause variance oscillations, as well.

6. CONCLUSION

In this paper we have performed IF estimation analysis based on the general quadratic shift-covariant class of TFD's. The exact bias and variance expressions are derived. It is shown that the IF estimation variance is closely related with the non-noisy signal's distribution. The expressions in the cases of most frequently used TFDs are obtained as special cases of the general analysis. The obtained results are proved numerically and statistically.

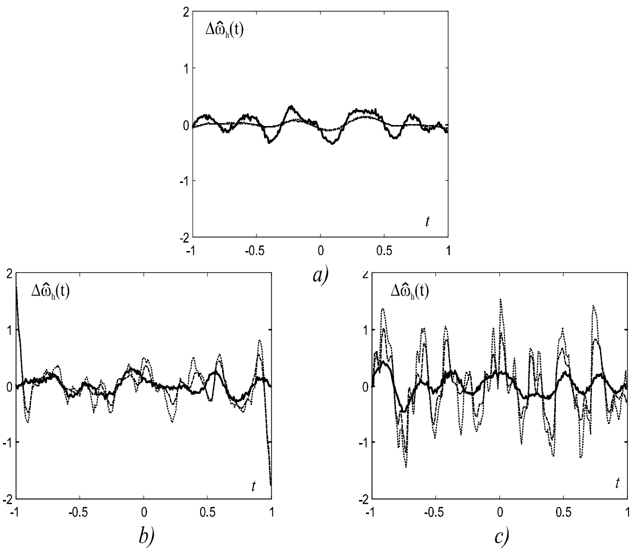

Fig.2. The IF estimation error in the cases of PWD (thick solid line), CWD (dashed line), BJD (dotted line), and for the different values of *a*: *a*) $a = 0$, *b*) $a = 0.25$, *c*) $a = 0.5$.

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REFERENCES

[1] M.G.Amin: “Minimum-variance time-frequency distribution kernels for signals in additive noise”, *IEEE Trans. on Signal Processing*, vol.44, no.9, Sept.1996, pp.2352-2356.

[2] B.Boashash: “Estimating and interpreting the instantaneous frequency of a signal - Part 1: Fundamentals”, *Proc. IEEE*, vol.80, Apr.1992, pp.519-538.

[3] L.Cohen: “Time-frequency distributions - a review”, *Proc. IEEE*, vol.77, July 1989, pp.941-981.

[4] L.Cohen: “Distributions concentrated along the instantaneous frequency”, *SPIE, Advanced Signal Processing Alg., Architect., and Implementations*, vol.1348, 1990, pp.149-157.

[5] L.Cohen, *Time-frequency analysis*, Prentice Hall, Englewood Cliffs, N.J., 1995.

[6] L.Cohen, C.Lee: “Instantaneous bandwidth”, in *Time-frequency signal analysis*, B. Boashash ed., Longman Cheshire, 1992.

[7] G.S.Cunningham, W.J.Williams: “Kernel decomposition of time-frequency distributions”, *IEEE Trans. on Signal Processing*, vol.42, June 1994, pp.1425-1441.

[8] A.Goldenshluger, A.Nemirovski: "On spatial adaptive estimation of nonparametric regression", *Res. report*, 5/94, Technion, Israel, Nov.1995.

[9] S.B.Hearon, M.G.Amin: "Minimum-variance time-frequency distribution kernels", *IEEE Trans. on Signal Processing*, vol.43, no.5, May 1995, pp.1258-1262.

[10] F.Hlawatsch, G.F.Boudreaux-Bartels: "Linear and quadratic time-frequency signal representation", *IEEE Signal Processing Magazine*, Apr.1992, pp.21-67.

[11] V.Ivanović, LJ.Stanković, D.Petranović: "Finite register length effects in implementation of distributions from the Cohen class", *IEEE Trans. on Signal Processing*, vol.46, no.7, July 1998, pp.2035-2041.

[12] V.Ivanović, M.Daković, LJ.Stanković: "Analysis of the instantaneous frequency estimation with the reduced interference distributions", *ETF Journal of Electrical Engineering*, vol.8, no.1, Nov.1999, pp.20-35.

[13] J.Jeong, W.J.Williams: "Kernel design for reduced interference distributions", *IEEE Trans. on Signal Processing*, vol.40, no.2, Feb.1992, pp.402-412.

[14] V.Katkovnik: "Local polynomial approximation of the instantaneous frequency: Asymptotic accuracy", *Signal Processing*, vol.52, no.3, 1996.

[15] V.Katkovnik: "Adaptive local polynomial periodogram for time-varying frequency estimation", in *Proc. of the IEEE Int. Symp. TFTSA*, Paris, June 1996, pp.329-332.

[16] V.Katkovnik: "Nonparametric estimation of instantaneous frequency", *IEEE Trans. on Information Theory*, vol.43, no.1, Jan.1997, pp.183-189.

[17] V.Katkovnik, LJ.Stanković: "Instantaneous frequency estimation using the periodogram with time-varying window length", *Signal Processing*, vol.67, no.3, June 1998, pp.345-358.

[18] V.Katkovnik, LJ.Stanković: "Instantaneous frequency estimation using the Wigner distribution with varying and data-driven window length", *IEEE Trans. on Signal Processing*, vol.46, no.9, Sept.1998, pp.2315-2326.

[19] S.M.Kay: "Statistically/computationally efficient frequency estimation", in *Proc. of the IEEE Int. Conf. ASSP*, New York, 1988, pp.2292-2295.

[20] B.C.Lovell, R.C.Williamson, B.Boashash: "The relationship between instantaneous frequency and time-frequency representations", *IEEE Trans. on Signal Processing*, vol.41, no.3, Mar.1993, pp.1458-1461.

[21] B.C.Lovell, R.C.Williamson: "The statistical performance of some instantaneous frequency estimators", *IEEE Trans. on Signal Processing*, vol.40, no.7, July 1992, pp.1707-1723.

[22] A.Papoulis, *Signal analysis*, McGraw-Hill Company, 1977.

[23] B.G.Quin: "Estimating frequency by interpolation using Fourier coefficients", *IEEE Trans. on Signal Processing*, vol.42, no.5, May 1994, pp.1264-1268.

[24] P.Rao, F.J.Taylor: "Estimation of the instantaneous frequency using the discrete Wigner distribution", *Electronics Letters*, vol.26, 1990, pp.246-248.

[25] LJ.Stanković: "Auto-term representation by the reduced interference distributions; A procedure for kernel design", *IEEE Trans. on Signal Processing*, vol.44, no.6, June 1996, pp.1557-1564.

[26] LJ.Stanković: "Analysis of noise in time-frequency distributions", *IEEE Trans. on Signal Processing*, submitted.

[27] LJ.Stanković, V.Ivanović: “Further results on the minimum variance time-frequency distribution kernels”, *IEEE Trans. on Signal Processing*, vol.45, no.6, June 1997, pp.1650-1655.

[28] LJ.Stanković, V.Ivanović, Z.Petrović: “Unified approach to the noise analysis in the Spectrogram and the Wigner distribution”, *Annales des Telecommunications*, no. 11/12, Nov/Dec.1996, pp.585-594.

[29] LJ.Stanković, V.Katkovnik: “Algorithm for the instantaneous frequency estimation using time-frequency distributions with adaptive window length”, *IEEE Signal Processing Letters*, vol.5, no.9, Sept.1998, pp.224-227.

[30] LJ.Stanković, V.Katkovnik: “Algorithm for the instantaneous frequency estimation using time-frequency distributions with adaptive window length”, *IEEE Signal Processing Letters*, vol.5, no.9, Sept.1998, pp.224-227.

[31] LJ.Stanković, S.Stanković: “On the Wigner distribution of discrete-time noisy signals with application to the study of quantization effects”, *IEEE Trans. on Signal Processing*, vol.42, no.7, July 1994, pp.1863-1867.

[32] K.M.Wong: “Estimation of the time-varying frequency of a signal: The Cramer-Rao bound and the application of Wigner distributions”, *IEEE Trans. on Acoust., Speech, Signal Processing*, vol.38, 3, 1990, pp.519-535.

[33] Y.Zhao, L.E.Atlas, R.J.Marks, II: “The use of cone-shaped kernels for generalized time-frequency representations on non-stationary signals”, *IEEE Trans., Acoust, Speech, Signal Processing*, vol.38, no.7, July 1990, pp.1084-1091.